FINAL FOR CALCULUS

Date: 2000, January 17, 1:10–3:00PM
Each problem is worth 10 points.

1.

(i) At time t, a particle has position

$$x(t) = 1 - \cos t, \qquad y(t) = t - \sin t.$$

Find the total distance traveled from $t = 0$ to $t = 2\pi$. Give the speed of the particle at $t = \pi$.

(ii) Find the area of the surface generated by revolving the curve $y = \cosh x$, $x \in [0, \ln 2]$ about the x-axis.

2.

(i) Find the tangent(s) to the curve

$$x(t) = -t + 2\cos\frac{1}{4}\pi t, \qquad Y(t) = t^4 - 4t^2$$

at the point $(2, 0)$.

(ii) Find the area of the region common to the circle $r = 2\sin\theta$ and the limaçon $r = \frac{3}{2} - \sin\theta$.

3.

(i) Verify that $\sinh 2t = 2\sinh t \cosh t$.

(ii) Compute $f'(x)$ where $f(x) = x^{2x}$.

4.

(i) Two years ago, there were 4 grams of a radioactive substance . Now there are 3 grams. How much was there 10 years ago?

(ii) Determine the exact values of $\sec^{-1}(-\sqrt{2})$ and $\sin^{-1}(\sin 7\pi/4)$.

5.

(i) Let f be continuous and define F by

$$F(x) = \int_0^x \left[t^2 \int_1^t f(u)\, du \right] dt.$$

Find $F'(x)$ and $F''(x)$.

(ii) Compute the limit

$$\lim_{x \to 0} \frac{1 - \cos 4x}{9x^2}.$$

6. Sketch the graph of the continuous function f that satisfies the conditions:
$f''(x) > 0$ if $|x| > 2$, $f''(x) < 0$ if $|x| < 2$;
$f'(0) = 0$, $f'(x) > 0$ if $x < 0$, $f'(x) < 0$ if $x > 0$;
$f(0) = 1$, $f(2) = \frac{1}{2}$, $f(x) > 0$ for all x, and f is an even function.

7. Evaluate the given integral

(i)

$$\int x(x+1)^9\, dx,$$

(ii)

$$\int \frac{\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta.$$

8.

(i) Evaluate the integral

$$\int \frac{dx}{e^x\sqrt{4+e^{2x}}}.$$

(ii) Show that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has no extreme values if and only if $a^2 < 3b$.

9. Show that, if u and v are differentiable functions of x and f is continuous, then

$$\frac{d}{dx}\left[\int_u^v f(t)\,dt\right] = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}.$$

Then compute

$$\frac{d}{dx}\left[\int_x^{x^2} \frac{dt}{t}\right].$$

10.

(i) Use mean value theorem to show that, if f is continuous on $[x, x+h]$ and differentiable on $(x, x+h)$, then

$$f(x+h) - f(x) = f'(x+\theta h)h$$

for some number θ between 0 and 1.

(ii) Let $h > 0$. Suppose that f is continuous on $[a-h, a+h]$ and differentiable on $(a-h, a)\cup(a, a+h)$. Use (i) to show that if

$$\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x) = L,$$

then f is differentiable at a and $f'(a) = L$.