MIDTERM 1 FOR ALGEBRA

Date: 2000, April 17, 15:10–17:00

Each of the following problems is worth 10 points.

1.

(i) Give the definition of a field.

(ii) Give an example of a unique factorization domain but not a principal ideal domain.

2.

(i) Give the definition of a vector space over a field F.

(ii) Give an example of an infinite-dimensional vector space over $\mathbf{R}$.

3.

(i) Construct a field of order 5.

(ii) Construct a field of order 25.

4. Find the greatest common divisor (in $\mathbf{Z}$) of 2178, 396, 792 and 726.

5.

(i) Give the definition of an algebraic closure of a field F.

(ii) Explain why $\mathbf{C}$ is not an algebraic closure of $\mathbf{Q}$.

6. Prove that if p is a prime in an integral domain D, then p is an irreducible.

7.

(i) Show that a field is a principal ideal domain.

(ii) Show that a field is a Euclidean domain.

8.

(i) What is $\mathbf{Z}[\sqrt{-5}]$?

(ii) Show that 7 is an irreducible in $\mathbf{Z}[\sqrt{-5}]$.

9. Show that if K is an algebraic extension of E and E is an algebraic extension of F, then K is an algebraic extension of F.

10.

(i) Find the degree and a basis of $\mathbf{Q}(\sqrt{2}, \sqrt{6})$ over $\mathbf{Q}(\sqrt{3})$.

(ii) Suppose that α is a transcendental number over $\mathbf{Q}$. Show that $1 + \alpha$ is also transcendental over $\mathbf{Q}$.