FINAL FOR ADVANCED LINEAR ALGEBRA

Date: Wednesday, January 17, 2001
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No credit will be given for an answer without reasoning.

1.

(i) [5%] Give an example of an algebra over $\mathbf{C}$ of dimension 5.

(ii) [5%] Let V be a three-dimensional vector space over $\mathbf{R}$. Give an example of a nonzero skew-symmetric bilinear form on V.

(iii) [5%] Give an example of a degree two representation of the group $\mathbf{Z}_3$.

2. [10%] Let V be an inner product space over $\mathbf{R}$. Suppose that S, T are subspaces of V such that V is the orthogonal direct sum of S and T. Show that the radical of V is the orthogonal direct sum of the radical of S and the radical of T i.e., prove that $\operatorname{rad}(V) = \operatorname{rad}(S) \perp \operatorname{rad}(T)$ if $V = S \perp T$.

3. [10%] Let V, W be vector spaces over a field F. Show that $V \otimes W$ and $W \otimes V$ are isomorphic.

4. [10%] Find an example of a bilinear map $\tau\colon V \times V \to W$ whose image $\operatorname{Im}(\tau) = \{\, \tau(u, v) \mid u, v \in V \,\}$ is not a subspace of W.

5. Let R be a commutative ring with identity.

(i) [5%] Show that any two nonzero elements in R are not linearly independent.

(ii) [5%] Using (i) conclude that an ideal I of R is a free R-module if and only if I is generated by an element of R that is not a zero divisor.

6. [10%] Suppose that V, W are vector spaces over a field F. Let $f\colon V \times V \to W$ be a map from $V \times V$ to W. Suppose that f is both linear and bilinear. Show that f is a zero map.

7. [10%] Let ρ denote the regular representation of the group $\mathbf{Z}_5$. Decompose ρ as a direct sum of irreducible representations.

8. [10%] Let $\rho_1\colon G_1 \to \mathrm{GL}(V_1)$ and $\rho_2\colon G_2 \to \mathrm{GL}(V_2)$ be representations of finite groups G_1, G_2 respectively. Define $\rho_1 \otimes \rho_2\colon G_1 \oplus G_2 \to \mathrm{GL}(V_1 \otimes V_2)$ by

$$(\rho_1 \otimes \rho_2)(g_1, g_2)\left(\sum_{\text{finite}} v_i \otimes w_i\right) = \sum_{\text{finite}} \rho_1(g_1)(v_i) \otimes \rho_2(g_2)(w_i)$$

for $g_1 \in G_1$, $g_2 \in G_2$, $v_i \in V_1$ and $w_i \in V_2$. Check that $\rho_1 \otimes \rho_2$ is a representation of $G_1 \oplus G_2$.

9. Let G be the group S_3 i.e., G is a non-abelian group of order 6 with elements $\{1, a, a^2, b, ba, ba^2\}$ and the relations $a^3 = 1$, $b^2 = 1$ and $ab = ba^2$. Let ρ be the regular representation of G, ρ_1 be the trivial representation of G. Define $\rho_2\colon G \to \mathbf{C}^*$ by defining $\rho_2(a) = 1$ and $\rho_2(b) = -1$.

(i) [5%] Check that ρ_2 can be made into a representation of G.

(ii) [5%] We know that there is a representation ρ_3 of G such that $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$. Is ρ_3 irreducible? Why or why not?

(iii) [5%] Find $\rho_3(a)$.