MIDTERM FOR ADVANCED CALCULUS

Time: 13:10–15:00, Monday, Apr 22, 2002
Instructor: Shu-Yen Pan

No credit will be given for an answer without reasoning.

1. Brief explanation is required for this problem.

(1) [5%] Give an example of a bounded function f such that $|f|$ is (Riemann) integrable on a bounded set A but f is not integrable on A.

(2) [5%] Give an example of a subset of $\mathbf{R}^n$ which has measure zero but whose boundary has positive measure.

2. [10%] Let A be an open set in $\mathbf{R}^n$ and f, g are differentiable function from A to $\mathbf{R}^m$. Prove that $f + g$ is also differentiable and

$$\mathbf{D}(f+g) = \mathbf{D}f + \mathbf{D}g.$$

3. [10%] A function $f \colon \mathbf{R}^n \to \mathbf{R}$ is called *homogeneous of degree* m if $f(tx) = t^m f(x)$ for all $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. If f is also differentiable, show that for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = m f(x).$$

(Hint: the chain rule)

4. [10%] Compute the second-order Taylor formula for $f(x, y) = e^x \cos y$ around $(0, 0)$.

5. [10%] Let $f(x) = x + 2x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) \neq 0$ but that f is not locally invertible near 0. Why does this not contradict the inverse function theorem?

6. [10%] Compute the index of $2x^2 + 6xy - y^2 - y^4$ at $(0, 0)$.

7. [10%] Find the extrema of $f(x, y, z) = x - y$ subject to the condition $x^2 - y^2 = 2$.

8. [10%] Evaluate $\lim_{n\to\infty} \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}}\, dx$.

9. [10%] Suppose that A is a bounded set in $\mathbf{R}$ and f is a bounded function on A. By the Lebesgue's theorem we know that if f is (Riemann) integrable on A, then the discontinuities of f have measure. Now we assume B is a subset of $\mathbf{R}$ (i.e., not necessarily bounded) and the function g is improper integrable on B. Show that the discontinuities of g also have measure zero.

10.

(1) [8%] Show that $\int_0^1 \ln x\, dx$ converges.

(2) [7%] Show that $\int_1^\infty \frac{1}{\ln x}\, dx$ diverges.