

MIDTERM FOR ADVANCED CALCULUS

Instructor: Shu-Yen Pan

Time: 13:10–15:00, Nov. 21, 2001

No credit will be given for an answer without reasoning.

1. [10%]
 - (i) Is the interval $[1, \infty) \subset \mathbf{R}$ closed? Why or why not?
 - (ii) The interval $[0, 1)$ is not compact. Give an open cover of $[0, 1)$ which does not have any finite subcover.
2. [10%] Let $f(x) = \sqrt{x}$.
 - (i) Is $f(x)$ uniformly continuous on $[0, 1]$? Why or why not?
 - (ii) Is $f(x)$ uniformly continuous on $[0, \infty)$? Why or why not?
 - (iii) Is a bounded continuous function necessarily uniformly continuous? Why or why not?
3. [10%] Let $A := \{(x, y) \in \mathbf{R}^2 \mid y = \sin \frac{1}{x}, x \neq 0\}$ regarded as a subset of \mathbf{R}^2 .
 - (i) Draw the picture of A on the xy -plane.
 - (ii) What is the interior of A ?
 - (iii) What is the closure of A ?
4. [10%] Let a be a real number. Suppose that $a \neq 0$. Show that $a^{-1} \neq 0$ and $(a^{-1})^{-1} = a$. (You may use the result: $0 \cdot x = 0$ for every $x \in \mathbf{R}^2$.)
5. [20%] Let d be the standard metric of \mathbf{R}^2 i.e.,
$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$
Define $\rho: \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}$ by
$$\rho((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$
 - (i) Check that (\mathbf{R}^2, ρ) is a metric space.
 - (ii) Draw the pictures of the open disk $D((1, 1), 1)$ for (\mathbf{R}^2, d) and for (\mathbf{R}^2, ρ) on the xy -plane.
 - (iii) Show that an open set in (\mathbf{R}^2, d) is also an open set in (\mathbf{R}^2, ρ) .
6. [10%] Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ defined by $(x, y) \mapsto x + y$. Prove from the definition (i.e., using ϵ - δ) that f is continuous.
7. [10%] Is it always true that $\text{bd}(A) = \text{bd}(\text{cl}(A))$? Why or why not?
8. [10%] Let A, B be two compact subsets in a metric space. Show that $A \cup B$ is also compact.
9. [10%] Suppose that A is a connected subset of \mathbf{R}^2 . Show that the closure of A is also connected.