FINAL FOR ADVANCED CALCULUS

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No credit will be given for an answer without reasoning.

1.

(i) Find a sequence a_n with $\limsup a_n = 5$ and $\liminf a_n = -3$.

(ii) Give an example of a contraction map $\Phi\colon \mathbf{R}^2 \to \mathbf{R}^2$ with fixed point $(1,1)$.

2. Let A, B be two non-empty subsets of $\mathbf{R}$.

(i) Is $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$? Prove it or give a counterexample.

(ii) Is $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$? Prove it or give a counterexample.

3. Suppose that $f\colon [0,1] \to [0,1]$ is continuous and onto. Show that there exists an $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

4. Suppose that $\sum_{k=1}^{\infty} a_k = \alpha\ (C,1)$ and $\sum_{k=1}^{\infty} b_k = \beta\ (C,1)$. Show that $\sum_{k=1}^{\infty}(a_k + b_k) = \alpha + \beta\ (C,1)$.

5. Let $f\colon A \to N$ be continuous and let $K \subseteq A$ be a compact set. Prove that f is uniformly continuous on K.

6. A subset A of $\mathbf{R}^2$ is called *convex* if $x, y \in A$ implies $tx + (1-t)y \in A$ for all $t \in [0,1]$. Show that a convex subset of $\mathbf{R}^2$ is connected.

7. Show that the following set

$$A = \{\, f \in \mathcal{C}([0,1], \mathbf{R}) \mid 0 \leq \int_0^1 f(x)\, dx \leq 3 \,\}$$

is closed in $\mathcal{C}([0,1], \mathbf{R})$.

8.

(i) Is the set A in problem 7 bounded? Why or Why not?

(ii) Let $f(x) = x^2 + 1$ and $g(x) = x$. Compute $d(f,g)$ in $\mathcal{C}([0,1], \mathbf{R})$.

9. Suppose that f is a differentiable function and α is a constant. Use definition to show that cf is also differentiable and $\mathbf{D}(\alpha f) = \alpha \mathbf{D} f$.

10. Suppose that $a_k \geq 0$ for all k and $\sum_{k=0}^{\infty} a_k = \alpha$. Prove that $\sum_{k=0}^{\infty} a_k x^k$ converges for $|x| < 1$ and $\lim_{x \to 1^-} \sum_{k=0}^{\infty} a_k x^k = \alpha$ by using the Weierstrass M-test.