L165100 - Fall 2010 - Homework 2

1. Prove that, for any Young diagram λ of n boxes, we have

$$\sum_{\lambda \lessdot \mu} f^{\mu} = (n+1) f^{\lambda},$$

where the sum is over all Young diagrams μ obtained by adding a box to λ.

2. Prove that, for any Young diagram λ, we have

$$\sum_{\lambda \lessdot \mu} c(\lambda, \mu) f^{\mu} = 0,$$

where $c(\lambda, \mu) = \mu_i - i$ if μ is obtained by adding a box to the i-th row of λ. (Hint: show that the linear map V on KY given by

$$V(\lambda) = \sum_{\lambda \lessdot \mu} c(\lambda, \mu) \mu$$

commutes with the usual lowering operator D on KY.)

3. Let P be a finite poset whose Hasse diagram is a rooted tree, that is, P has a unique minimal elements $\hat{0}$, and, for every $x \in P$, the interval $[\hat{0}, x]$ is a chain. Show that the number N of linear extensions of P (that is, the number of total orderings of P compatible with the given partial order) is given by the following hook-length type formula: $N = |P|!/\Pi_{x \in P} h(x)$, where $h(x)$ is the number of elements $y \in P$ such that $x \leq y$.