Linear Algebra II, Midterm, Yung-fu Fang, 2010/11/09

Show All Work

1. (a) State the test for diagonalization. [5%]
(b) State the Cayley-Hamilton Theorem. [5%]
(c) State the Gram-Schmidt Process. [5%]
(d) State the Schur Theorem. [5%]

2. Let $T : P_2(R) \to P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Show that T is diagonalizable and find the matrices Q and D such that $Q^{-1}AQ = D$. [10%]

3. Let $T : R^2 \to R^2$ be the rotation by θ. Prove that T is a linear operator. Is T diagonalizable? Explain! [10%]

4. Let $B_1 \in M_{k\times k}(F)$, $B_2 \in M_{k\times(n-k)}(F)$, and $B_3 \in M_{(n-k)\times(n-k)}(F)$. Show that [10%]

$$\det \begin{pmatrix} B_1 - tI_k & B_2 \\ 0 & B_3 - tI_{n-k} \end{pmatrix} = \det\left(B_1 - tI_k\right)\det\left(B_3 - tI_{n-k}\right).$$

5. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Define $\overline{T} : V/W \to V/W$ by $\overline{T}(v+W) = T(v) + W$ for any $v + W \in V/W$. Show that if both T_W and $\overline{T}$ are diagonalizable and have no common eigenvalues, then T is diagonalizable. [10%]

6. Let V be a finite-dimensional inner product space with an orthonormal ordered basis $\beta = \{v_1, \cdots, v_n\}$, T a linear operator on V, and the matrix $A = [T]_\beta$. Prove that, for all i and j, $A_{ij} = < T(v_j), v_i >$. Give a direct proof. [10%]

7. Let $\|\cdot\|$ be a norm on a real vector space V satisfying the parallelogram law,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Define

$$< x, y > = \frac{1}{4}\Big[\|x+y\|^2 - \|x-y\|^2\Big].$$

Show that

(a) $< x, 2y > = 2 < x, y >$, for all $x, y \in V$. [5%]
(b) $< x+u, y > = < x, y > + < u, y >$, for all $x, u, y \in V$. [5%]

8. Let $A \in M_{m\times n}(F)$ and $b \in F^m$. Suppose that the system of equations $Ax = b$ is consistent.
(a) Prove that $R(L_{A^*})^\perp = N(L_A)$. [5%]
(b) Prove that the minimal solution s to $Ax = b$ is in $R(L_{A^*})$. [5%]
(c) Find the minimal solution to $\begin{cases} x + 2y - z = 1, \\ 2x + 3y + z = 2, \\ 4x + 7y - z = 4. \end{cases}$ [5%]

9. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Show that V has an orthonormal basis of eigenvectors of T. Hence that T is self-adjoint. [10%]